On the Structure and Invariants of Endomorphism Graphs of Finite Groups

Extended Abstract submitted for consideration for the Best Paper Presentation Award

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1 Introduction

The study of graphs defined on algebraic structures, particularly groups, has become an important area of interaction between algebra and graph theory. Classical constructions such as Cayley graphs trace their origins to the nineteenth century and continue to play a central role in algebraic combinatorics and geometric group theory. Beyond Cayley graphs, several graph structures defined purely through intrinsic group properties have been introduced and studied. A notable early example is the commuting graph, introduced by Brauer and Fowler, where vertices represent group elements and edges correspond to commuting pairs. This construction found use in their groundbreaking arguments towards the classification of finite simple groups. Other graphs such as the generating graph and the power graph have been developed to investigate algebraic properties through graph-theoretic behavior. Motivated by this rich interaction, the present study focuses on graphs built from the endomorphisms of group G. Specifically, we study the directed endomorphism graph, where an edge from vertex a to b exists whenever there is an endomorphism of G mapping a to b; the corresponding undirected endomorphism graph is the underlying simple graph. These constructions yield new ways of examining structural properties of groups and provide insight into how algebraic mappings influence combinatorial representations. Situating these graphs within the broader framework of transformation monoids allows us to connect them to extract significant structural information. This work aims to systematically analyze these graphs, compute them for important classes of finite groups, and understand the ways in which algebraic and graph-theoretic properties interact.

1.1 Directed and undirected endomorphism graph

An **endomorphism** of a group G is a map $f : G \to G$ such that $(x * y)^f = x^f * y^f$. The **endomorphism digraph** $\overrightarrow{\mathrm{EG}}(G)$ of a group G takes the vertex set to be G with an arc from x to y if some endomorphism of G maps x to y. The **endomorphism graph** $\mathrm{EG}(G)$ is obtained by ignoring the directions and suppressing double edges that result. There are **compressed** versions of these, obtained by deleting the identity and shrinking each automorphism class to a single vertex. We denote these by $\overrightarrow{\mathrm{EG}}_{-}(G)$ and $\mathrm{EG}_{-}(G)$ respectively.

These graphs are studied using the framework of transformation monoids, which clarifies adjacency relations and structural behavior. $\overrightarrow{\mathrm{EG}}(G)$ is a digraph attached to a transformation monoid and hence $\mathrm{EG}(G)$ is a **perfect graph**.

2 Results

- **Isomorphism**
  An isomorphism between compressed endomorphism digraphs $\overrightarrow{\mathrm{EG}}_{-}(G)$ and $\overrightarrow{\mathrm{EG}}_{-}(H)$ is said to be **strong** if it maps each automorphism class in G to an automorphism class of the same size in H.

  - ▷ The directed endomorphism graphs (and hence the endomorphism graphs) of isomorphic groups are isomorphic.
  - ▷ Can there be an isomorphism from $\overrightarrow{\mathrm{EG}}_{-}(G)$ to $\overrightarrow{\mathrm{EG}}_{-}(H)$ which is not strong? Cyclic groups of different prime orders.
  - ▷ Can non-isomorphic groups have isomorphic directed endomorphism graphs? Cyclic group Z_{p^3},the group $Z_{p^2} \times Z_p$, and the non-abelian group of order p^3 and exponent p^2, where p is an odd prime.

- **Direct Products**
  The **strong product** of two digraphs (or graphs) Γ and Δ, $\Gamma \boxtimes \Delta$ on vertex sets X and Y is the graph whose vertex set is the Cartesian product $X \times Y$, with an arc (or edge) from (x, y) to (x', y') if one of the following holds:

  (a) $x \to x'$ (or $x \sim x'$) in Γ, $y = y'$;
  (b) $x = x'$, $y \to y'$ (or $y \sim y'$) in Δ;
  (c) $x \to x'$ (or $x \sim x'$) in Γ, $y \to y'$ (or $y \sim y'$) in Δ.

  Let G and H be groups with coprime orders. Then the endomorphism monoid of $G \times H$ is $\mathrm{End}(G) \times \mathrm{End}(H)$, and the endomorphism digraph of $G \times H$ is the strong product of $\overrightarrow{\mathrm{EG}}(G)$ and $\overrightarrow{\mathrm{EG}}(H)$. The procedure of collapsing automorphism classes commutes with this isomorphism, but it does not follow that $\overrightarrow{\mathrm{EG}}_{-}(G \times H)$ is isomorphic to $\overrightarrow{\mathrm{EG}}_{-}(G) \boxtimes \overrightarrow{\mathrm{EG}}_{-}(H)$.

- **Cyclic Groups**
  There is an endomorphism mapping x to y in $\mathbb{Z}_n$ if and only if $\gcd(x, n)$ divides $\gcd(y, n)$ if and only if there is an automorphism mapping x to y. The compressed endomorphism digraph has vertices indexed by divisors of n (except 1), with an

edge $[x] \to [y]$ whenever x divides y. In particular, $\mathrm{EG}_{-}(Z_n)$ is complete if and only if n is a prime power.

- ▷ Let G be a cyclic group of order n and $1 < d_1 \leq d_2 \leq \ldots \leq d_k < n$ be the divisors of n. Then the total number of edges in $\mathrm{EG}(G)$ is

$$\binom{n}{2} - \sum_{\substack{1 \leq i < j \leq k \\ d_i \nmid d_j}} \phi(d_i)\phi(d_j). \tag{1}$$

- ▷ The number of maximal cliques in $\mathrm{EG}(\mathbb{Z}_n)$, where $n = {p_1}^{n_1}{p_2}^{n_2} \ldots {p_k}^{n_k}$ in which $p_1, p_2, \ldots, p_k$ are distinct primes and $n_i \in \mathbb{N}$, for every $i \in \{1, 2, \ldots, k\}$ is

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1! n_2! \cdots n_k!}$$

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- ▷ Let $n = p_1^{n_1} \cdots p_k^{n_k}$, where $p_1, \ldots, p_k$ are distinct primes and $n_1, \ldots, n_k$ positive integers. Write the primes $p_1, \ldots, p_k$ (with multiplicities $n_1, \ldots, n_k$) in a sequence of length $n_1 + \cdots + n_k$ where the primes are in nonincreasing order, say $(q_1, q_2, \ldots, q_m)$, where $m = n_1 + \cdots + n_k$, and let r_i be the product of the first i terms in this sequence (with $r_0 = 1$). Then the clique number and chromatic number of $\mathrm{EG}(Z_n)$ are equal to $\sum_{i=0}^{m} \phi(r_i)$.

- **Power Graph**
  The **directed power graph** $\vec{P}(G)$ of a finite group G has vertex set G, with an arc $x \to y$ whenever y is a power of x. $\overrightarrow{\mathrm{EG}}(Z_n)$ is the same as $\vec{P}(Z_n)$.

  - ▷ If G is a finite group, then the **power graph** $\overrightarrow{P}(G)$ is a spanning subgraph of $\overrightarrow{\mathrm{EG}}(G)$. Equality holds if and only if G is cyclic.

- **Abelian Groups**

  - ▷ The endomorphism graph of an abelian group G is **complete** if and only if $G = (\mathbb{Z}_{p^a})^m \times (\mathbb{Z}_{p^{a+1}})^n$ for some $m, n \geq 0, a \geq 1$.
  - ▷ For $a, b \in G$(finite and abelian), $|b|$ divides $|a|$ if and only if $G \simeq \prod_{i=1}^{k} (\mathbb{Z}_{{p_i}^{n_i}})^{m_i}$ for distinct primes p_i and $n_i, m_i \in \mathbb{N}$.
  - ▷ A **point basis** in a digraph is a set such that every vertex receive an incoming arc from at least one vertex in the set (or be part of the set itself). If G is an abelian group, then $\overrightarrow{\mathrm{EG}}(G)$ has **point basis** with cardinality 1.
  - ▷ If there exists an $a \in G$ such that $|G : C(a)| > 3$, where $C(a)$ denotes the centralizer of a, then $\mathrm{EG}(G)$ is non-planar. If G is an abelian group, $\mathrm{EG}(G)$ is **planar** if and only if $|G| \leq 4$.
  - ▷ The endomorphism graph of a group G is a **tree** if and only if $G = \mathbb{Z}_2$.

- Structures of compressed endomorphism digraph of $\mathbf{D_{2n}}, \mathbf{Dic_n}, \mathbf{S_n}, \mathbf{A_n}, \mathbb{Z}_\mathbf{q} \rtimes \mathbb{Z}_\mathbf{m}$, abelian p-groups.

- **Identity element deleted subgraph**
  In the case of directed endomorphism graphs there is a directed arc to e from all other vertices. So the presence of vertex corresponding to identity element is not interesting and hence it is a common practice to consider the graph induced by group elements other than the identity element. $\overrightarrow{\mathrm{EG}^*}(G)$ and $\mathrm{EG}^*(G)$, denotes the induced subgraph obtained from $\overrightarrow{\mathrm{EG}}(G)$ and $\mathrm{EG}(G)$, respectively, by deleting the vertex corresponding to the identity element.

  ▷ For $\overrightarrow{\mathrm{EG}^*}(G)$ the following are equivalent:

  (i) $\overrightarrow{\mathrm{EG}^*}(G)$ is diconnected.

  (ii) $\overrightarrow{\mathrm{EG}^*}(G)$ is a complete digraph.

  (iii) $\overrightarrow{\mathrm{EG}^*}(G)$ is Hamiltonian.

  ▷ $\overrightarrow{\mathrm{EG}}(G)$ is not diconnected, since there is no endomorphism mapping the identity element to any other element. Let G be an abelian group. Then $\overrightarrow{\mathrm{EG}}(G^*)$ $\Big(G^* = G - \{e\}\Big)$ is **diconnected** or **Hamiltonian** if and only if $G \simeq (\mathbb{Z}_p)^k$ under addition $\underbrace{(+_p, +_p, \ldots, +_p)}_{k}$, for some prime p and $k \in \mathbb{N}$.

  ▷ Given a group G, $\mathrm{EG}^*(G)$ is a tree if and only if $G = \mathbb{Z}_2$ or $\mathbb{Z}_3$ under $+_2$ and $+_3$, respectively.

3 Conclusion

In this work, we studied graphs defined using endomorphisms of finite groups, and showed how the underlying group structure is reflected in the properties of the associated graphs. General results were obtained along with explicit examples for several well-known families of finite groups. We also demonstrated that these graphs extend and generalize existing constructions, such as power graphs, and answered questions related to the isomorphism of endomorphism graphs. As future work, we plan to investigate additional graph invariants and explore further families of finite groups that remain unexplored in this context.

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